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Flux of Momentum of N-Bodies System by Gravitational Waves

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Abstract

The energy and angular momentum carried by gravitational waves of an N-body system has been extensively studied by the author. In this paper the linear momentum, within general relativity, is investigated by studying waves emitted from a source consisting of N-particles moving under their own gravitation.

1. *Introduction*

We assume that the gravitational field is weak everywhere.

The masses are considered as point-like, so that the mass density μ is defined by

$$\mu(\mathbf{x}, t) = \sum_{\nu=1}^{m} m_\nu \delta(\mathbf{x} - \mathbf{x}_\nu(t)) \tag{1.1}$$

To avoid the singularities δ is a *good-delta function* having the properties (Infeld & Plebànski, 1960)

$$\int_V f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}')\, d\mathbf{x} = f(\mathbf{x}') \tag{1.2}$$

$$\int_V \frac{f(\mathbf{x})}{|\mathbf{x} - \mathbf{x}'|^p} \delta(\mathbf{x} - \mathbf{x}')\, d\mathbf{x} = 0, \qquad \text{for } p = 1, 2, 3, \ldots \tag{1.3}$$

where $f(\mathbf{x})$ is regular at $\mathbf{x} = \mathbf{x}'$. Also, $\mathbf{x}$ = the field point, $\mathbf{x}'$ = the source points and m_ν the masses of the particles ($\mathbf{x} = (x^1, x^2, x^3)$).

The N-body system is assumed to be bounded and isolated in the sense that it may radiate but no radiation enters it.

2. *The Flux of Momentum*

We consider the distribution of matter of N-bodies described by a symmetric complex energy-momentum tensor (Landau & Lifshitz, 1971)

$$\Theta^{ik} = -g(T^{ik} + t^{ik}) \tag{2.1}$$

where g is the determinant of the metric tensor coefficients g_{ik}, T^{ik} the energy-momentum tensor and t^{ik} the pseudo tensor. (Latin indices take the values 0, 1, 2, 3. Greek indices take the values 1, 2, 3, and $x^0 = ct$, where c and t denote the speed of light and the time.)

Equation (2.1) in the Newtonian approximation becomes

$$\Theta_0^{ik} = -g(T^{ik} + t^{ik})_0 \tag{2.2}$$

where the index 0 means Newtonian approximation.

The dominant terms of equations (2.2) are (Dionysiou, 1973)

$$\Theta_0^{00} = c^2 \sum_{m'} m'\delta(\mathbf{x} - \mathbf{x}') \tag{2.3}$$

$$\Theta_0^{0\alpha} = c \sum_{m'} m'u'_\alpha\delta(\mathbf{x} - \mathbf{x}') \tag{2.4}$$

$$\Theta_0^{\alpha\beta} = \sum_{m'} m'u'_\alpha u'_\beta\delta(\mathbf{x} - \mathbf{x}') + t^{\alpha\beta} \tag{2.5}$$

and

$$\Theta_0'^{\alpha\beta} = \int \Theta_0^{\alpha\beta}\, d\mathbf{x}' \tag{2.6}$$

where

$$\Theta_0'^{\alpha\beta} = \sum_m m u_\alpha u_\beta + t'^{\alpha\beta} \tag{2.7}$$

and

$$t^{\alpha\beta} = -\tfrac{1}{2}G\sum_{m'}\sum_{m''}\frac{m'm''(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|\mathbf{x} - \mathbf{x}'|^3}\,\delta(\mathbf{x} - \mathbf{x}'') \qquad (\text{mod. div.}) \tag{2.8}$$

where G denotes Newton's constant.

From equation (2.7), we have defined (Dionysiou, 1973)

$$t'^{\alpha\beta} = \int t^{\alpha\beta}(\mathbf{x}, t)\, d\mathbf{x} \tag{2.9}$$

and it follows

$$t'^{\alpha\beta} = -\tfrac{1}{2}G\sum_{m'}\sum_{m''}\frac{m'm''(x''_\alpha - x'_\alpha)(x''_\beta - x'_\beta)}{|\mathbf{x}'' - \mathbf{x}'|^3} \qquad (\text{mod. div.}) \tag{2.10}$$

The field equations are

$$\frac{\partial^2}{\partial x^l \partial x^m}[(-g)(g^{ik}g^{lm} - g^{il}g^{km})] = \frac{16\pi G}{c^4}(-g)(T^{ik} + t^{ik}) \qquad (2.11)$$

Since the left-hand side is antisymmetric in (i, m) and (k, l), it follows that

$$\Theta^{ik}{}_{,k} = \Theta^{ik}{}_{,i} = 0 \qquad (2.12)$$

where $,k$ means $\partial/\partial x^k$.

Using equation (2.2) and the definition

$$\sqrt{(-g)}g^{ik} = n^{ik} + \gamma^{ik} \qquad (2.13)$$

where n^{ik} is the Galilean metric, we get that

$$\Box \gamma^{ik} = \frac{16\pi G}{c^4}\Theta_0^{ik} \qquad (2.14)$$

where

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

From equations (2.12) and (2.14) we have

$$\gamma^{ik}{}_{,k} = \gamma^{ik}{}_{,i} = 0 \qquad (2.15)$$

i.e. the de Donder harmonic coordinate condition.

The general solution of equation (2.14) consists of a mixture of advanced and retarded potentials plus any solution of the free field equations

$$\Box \gamma^{ik} = 0 \qquad (2.16)$$

but on physical grounds we are interested in the retarded potentials, which satisfy the outgoing radiation condition, i.e. the sources must be sources, not sinks, of momentum so that

$$\gamma^{ik}(\mathbf{x}, t) = \frac{4G}{c^4}\int \frac{\Theta_0^{ik}\left[\mathbf{x}', t - \dfrac{|\mathbf{x} - \mathbf{x}'|}{c}\right]}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' \qquad (2.17)$$

where the integration has to be effected over the whole of the three-dimensional space, since Θ^{ik}, unlike T^{ik}, need not necessarily vanish outside the volume occupied by the system of N-bodies. From equation (2.12), we take

$$\Theta_0^{ik}{}_{,k} = \Theta_0^{ik}{}_{,i} = 0 \qquad (2.18)$$

which are the conservation laws of the system.

The first terms of an expansion of the integrand of equation (2.17) in powers of $\mathbf{n} \cdot \mathbf{x}'$ give

$$\gamma^{ik}(\mathbf{x}, t) = \frac{4G}{c^4 r}\left[\int \Theta_0^{ik}\left(\mathbf{x}', t - \frac{r}{c}\right) d\mathbf{x}' + n_\alpha c^{-1}\frac{\partial}{\partial t}\int \Theta_0^{ik} x'^\alpha \, d\mathbf{x}'\right] \qquad (2.17a)$$

where $|\mathbf{x}-\mathbf{x}'| = r - \mathbf{n}\cdot\mathbf{x}' + O(r^{-1})$, $r = |\mathbf{x}|$ and $\mathbf{n} = \mathbf{x}/r$.

Then from equation (2.17a) and using Bekenstein (1973) putting Θ^{ik}, where T^{ik}, we obtain a similar result for the linear momentum in lowest order as a quadrupole–octopole cooperative effect by the equation

$$P^{\alpha} = \frac{G}{945c^6}\,[22Q^{\beta\gamma}B^{\beta\gamma\alpha} - 12Q^{\beta\gamma}B^{\beta\alpha\gamma} - 12Q^{\beta\alpha}B^{\beta\gamma\gamma}] \tag{2.19}$$

where P^{α} is the total outflux per unit time of the αth component of linear momentum of N-bodies system,

$$Q^{\alpha\beta} = \frac{\partial^3}{\partial t^3}\int \Theta_0^{00}c^{-2}(3x'^{\alpha}x'^{\beta} - \delta_{\alpha\beta}|\mathbf{x}'|^2)\,d\mathbf{x}' \tag{2.20}$$

$$B^{\alpha\beta\nu} = \frac{1}{c}\,(\tfrac{1}{5}L^{\alpha\beta\nu} - 2N^{\alpha\beta\nu}) \tag{2.21}$$

$$L^{\alpha\beta\nu} = \frac{\partial^4}{\partial t^4}\int \Theta_0^{00}c^{-2}(5x'^{\alpha}x'^{\beta} - \tfrac{5}{3}\delta_{\alpha\beta}|\mathbf{x}'|^2)x'^{\nu}\,d\mathbf{x}' \tag{2.22}$$

$$N^{\alpha\beta\nu} = \frac{\partial^3}{\partial t^3}\int c^{-1}(K^{0\nu\beta}x'^{\alpha} + K^{0\nu\alpha}x'^{\beta} - \tfrac{2}{3}\delta_{\alpha\beta}K^{0\nu\delta}x'^{\delta})\,d\mathbf{x}' \tag{2.23}$$

and we have introduced the auxiliary angular momentum tensor

$$K^{ijk} = \Theta^{ij}x'^{k} - \Theta^{ik}x'^{j} \tag{2.24}$$

The three tensors $Q^{\alpha\beta}$, $L^{\alpha\beta\nu}$, $N^{\alpha\beta\nu}$ and $B^{\alpha\beta\nu}$ are all symmetric and traceless in their first two indices.

We define

$$I_{\alpha\beta} = \sum_m m x^{\alpha}x^{\beta} \tag{2.25}$$

as the moment of inertia tensor. Also,

$$I_{\alpha\beta} = I_{\beta\alpha} \tag{2.26}$$

From equation (2.25), we have

$$\frac{1}{2}\frac{d}{dt}I_{\alpha\beta} = \sum_m m u^{\alpha}x^{\beta} = \sum_m m x^{\alpha}u^{\beta} \tag{2.27}$$

We define

$$D_{\alpha\beta} = \sum_m m x^{\alpha}x^{\beta} - \tfrac{1}{3}\delta_{\alpha\beta}\sum_m m x^{\gamma}x^{\gamma} \tag{2.28}$$

as the quadrupole tensor of the system of particles. Also,

$$D_{\alpha\beta} = D_{\beta\alpha} \tag{2.29}$$

From equations (2.3), (2.20) and (2.29), we obtain

$$Q^{\alpha\beta} = \frac{\partial^3}{\partial t^3} \int c^2 \sum_{m'} m' \delta(\mathbf{x} - \mathbf{x}') c^{-2} (3x'^\alpha x'^\beta - \delta_{\alpha\beta} |x'|^2)\, d\mathbf{x}'$$

$$= \frac{\partial^3}{\partial t^3} \sum_m m(3x^\alpha x^\beta - \delta_{\alpha\beta} |\mathbf{x}|^2) = 3\, \frac{d^3 D_{\alpha\beta}}{dt^3} = 3\dddot{D}_{\alpha\beta} \tag{2.30}$$

Also, using equations (2.3), (2.4), (2.21), (2.22), (2.23) and (2.24), we obtain

$$\begin{aligned} B^{\alpha\beta\nu} = \frac{1}{c} \Bigg\{ & \frac{1}{5} \frac{\partial^4}{\partial t^4} \int \sum_{m'} m' (5x'^\alpha x'^\beta - \tfrac{5}{3} \delta_{\alpha\beta} |\mathbf{x}'|^2) x'^\nu \, d\mathbf{x}' \\ & - \frac{2}{c} \frac{\partial^3}{\partial t^3} \int [(\Theta^{0\nu} x'^\beta x'^\alpha - \Theta^{0\beta} x'^\nu x'^\alpha) + (\Theta^{0\nu} x'^\alpha x'^\beta \\ & - \Theta^{0\alpha} x'^\nu x'^\beta) - \tfrac{2}{3} \delta_{\alpha\beta} (\Theta^{0\nu} x'^\delta x'^\delta - \Theta^{0\delta} x'^\nu x'^\delta)]\, d\mathbf{x}' \Bigg\} \end{aligned} \tag{2.31}$$

Integrating equation (2.31), we obtain

$$\begin{aligned} B^{\alpha\beta\nu} = \frac{1}{c} \Bigg\{ & \frac{\partial^4}{\partial t^4} (D_{\alpha\beta} x^\nu) - 2 \frac{\partial^3}{\partial t^3} \Bigg[\Bigg(\sum_m m u^\nu x^\beta - \sum_m m u^\beta x^\nu \Bigg) x^\alpha \\ & + \Bigg(\sum_m m u^\nu x^\alpha - \sum_m m u^\alpha x^\nu \Bigg) x^\beta - \tfrac{2}{3} \delta_{\alpha\beta} \Bigg(\sum_m m u^\nu x^\delta \\ & - \sum_m m u^\delta x^\nu \Bigg) x^\delta \Bigg] \Bigg\} \end{aligned} \tag{2.32}$$

and using equation (2.27), equation (2.32) becomes

$$B^{\alpha\beta\nu} = \frac{1}{c} \frac{\partial^4}{\partial t^4} (D_{\alpha\beta} x^\nu) \tag{2.33}$$

Putting equations (2.30) and (2.33) into equation (2.19) we get that

$$\begin{aligned} P^\alpha = \frac{G}{315c^7} \Bigg[& 22\dddot{D}_{\beta\gamma} \frac{\partial^4}{\partial t^4} (D_{\beta\gamma} x^\alpha) - 12\dddot{D}_{\beta\gamma} \frac{\partial^4}{\partial t^4} (D_{\alpha\beta} x^\gamma) \\ & - 12\dddot{D}_{\alpha\beta} \frac{\partial^4}{\partial t^4} (D_{\beta\gamma} x^\gamma) \Bigg] \end{aligned} \tag{2.34}$$

which is the linear momentum radiated by the N-bodies moving under their own gravitation.

Now, we suppose the z-axis as the axis of symmetry of N-bodies system and θ the polar angle measured from it. Then, we get that

$$D_{12} = D_{13} = D_{23} = 0 \tag{2.35}$$

from equation (2.29) and

$$\dddot{D}_{12} = \dddot{D}_{13} = \dddot{D}_{23} = 0 \tag{2.36}$$

We note that $D_{\alpha\beta}$ is a function of time only. From the traceless of $D_{\alpha\beta}$,

$$D_{\alpha\alpha} = 0 \tag{2.37}$$

and $D_{\alpha\beta} = D_{\beta\alpha}$, it follows

$$\dddot{D}_{11} = \dddot{D}_{22} = -\tfrac{1}{2}\dddot{D}_{33} \tag{2.38}$$

Taking these results into account in equation (2.34), one finds (we put $x = x^1$, $y = x^2$, $z = x^3$)

$$P^3 = \frac{G}{35c^7}\dddot{D}_{33}\frac{\partial^4}{\partial t^4}(D_{33}x^3) \tag{2.39}$$

where P^1 and P^2 must vanish by symmetry.

Using Bekenstein (1973), we take

$$\dddot{D}_{11} = \dddot{D}_{22} = -\tfrac{1}{2}\dddot{D}_{33} = -\frac{1}{3}\frac{\partial^3}{\partial t^3}q_2 \tag{2.40}$$

and since, from equation (2.22),

$$3L^{333} - 4L^{311} = 10\frac{\partial^4}{\partial t^4}q_3 \tag{2.41}$$

or

$$15\frac{\partial^4}{\partial t^4}(D_{33}x^3) = 10\frac{\partial^4}{\partial t^4}q_3 \tag{2.42}$$

it follows

$$q_3 = \tfrac{3}{2}D_{33}x^3 \tag{2.43}$$

Putting equations (2.40) and (2.43) into (2.39), we obtain

$$P^3 = \frac{4G}{315c^7}\cdot\frac{\partial^3 q_2}{\partial t^3}\cdot\frac{\partial^4 q_3}{\partial t^4} \tag{2.44}$$

where

$$q_n = \int \Theta_0^{0} c^{-2} P_n(\cos\theta)|\mathbf{x}'|^n\, d\mathbf{x}', \qquad n = 2, 3 \tag{2.45}$$

and $P_n(\cos\theta)$ is the nth Legendre polynomial (Bekenstein, 1973).

Equation (2.39) is a result on axially symmetric transport of momentum by gravitational waves. Bonnor & Rotenberg (1961) have obtained a similar result for a special model of two particles.

Thus according to equations (2.34) and (2.39) the gravitational waves remove momentum from the N-bodies particle sources.

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Appendix

I. The energy-momentum tensor of the matter is

$$T^{ik} = \sum_{\nu=1}^{n} \frac{m_\nu c}{\sqrt{(-g)}} \cdot \frac{dx^i}{ds} \cdot \frac{dx^k}{dt} \cdot \delta(\mathbf{x} - \mathbf{x}_\nu) \tag{A.1}$$

from which follow (Dionysiou, 1973)

$$T^{00} = \sum_{\nu=1}^{n} m_\nu c^2 \delta(\mathbf{x} - \mathbf{x}_\nu) + O(1) \tag{A.2}$$

$$T^{0\alpha} = \sum_{\nu=1}^{n} m_\nu c u_\alpha \delta(\mathbf{x} - \mathbf{x}_\nu) + O(c^{-1}) \tag{A.3}$$

and

$$T^{\alpha\beta} = \sum_{\nu=1}^{n} m_\nu u_\alpha u_\beta \delta(\mathbf{x} - \mathbf{x}_\nu) + O(c^{-2}) \tag{A.4}$$

II. The pseudo-energy-momentum tensor (Chandrasekhar & Esposito, 1970) is

$$t^{00} = -\frac{7}{8\pi G}\left(\frac{\partial V}{\partial x_\gamma}\right)^2 + O(c^{-2}) \tag{A.5}$$

$$t^{0\alpha} = 0 + O(c^{-1}) \tag{A.6}$$

and

$$t^{\alpha\beta} = \frac{1}{16\pi G}\left[4\frac{\partial V}{\partial x_\alpha}\frac{\partial V}{\partial x_\beta} - 2\delta_{\alpha\beta}\left(\frac{\partial V}{\partial x_\gamma}\right)^2\right] + O(c^{-2}) \tag{A.7}$$

where

$$4\frac{\partial V}{\partial x_\alpha}\frac{\partial V}{\partial x_\beta} - 2\delta_{\alpha\beta}\left(\frac{\partial V}{\partial x_\gamma}\right)^2 = -8\pi G\mu\frac{\partial^2 V^*}{\partial x_\alpha \partial x_\beta} - 8\pi G\mu V\delta_{\alpha\beta} \qquad \text{(mod. div.)}$$

$$= -8\pi G\mu\left(\frac{\partial^2 V^*}{\partial x_\alpha \partial x_\beta} + V\delta_{\alpha\beta}\right)$$

$$= -8\pi G^2\mu\sum_{m'}\frac{m'(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|\mathbf{x} - \mathbf{x}'|^3}$$

and

$$V^* = -G \sum_{m'} m' |\mathbf{x} - \mathbf{x}'|$$
$$\mu = \sum_{m'} m' \delta(\mathbf{x} - \mathbf{x}') = \sum_{m''} m'' \delta(\mathbf{x} - \mathbf{x}'')$$

therefore from equation (A.7) follows

$$t^{\alpha\beta} = -\tfrac{1}{2} G \sum_{m'} \sum_{m''} \frac{m' m'' (x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|\mathbf{x} - \mathbf{x}|^3} \delta(\mathbf{x} - \mathbf{x}'') \qquad \text{(mod. div.)} \qquad \text{(A.8)}$$

III. We have put (Chandrasekhar & Esposito, 1970)

$$f(\mathbf{x}, t) \equiv g(\mathbf{x}, t) \qquad \text{(mod. div.)} \qquad \text{(A.9)}$$

if the functions f, g differ by the divergence of a vector, which vanishes sufficiently rapidly at infinity that their integrals over the whole of space (assuming that they exist) are equal.

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